Numerical resolution of semi-discrete generated Jacobian equation and application to non-imaging optics

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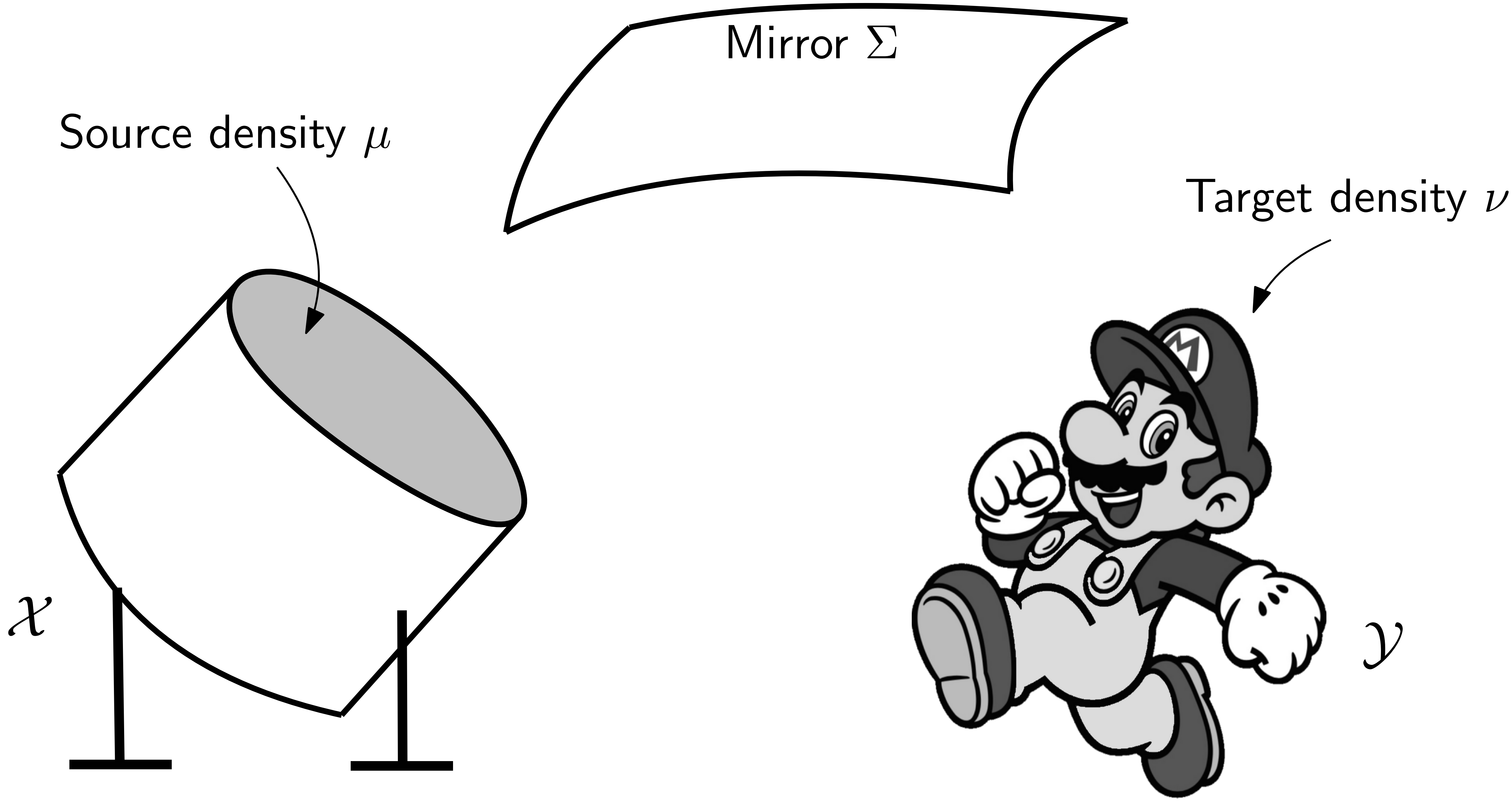
Non-imaging optics

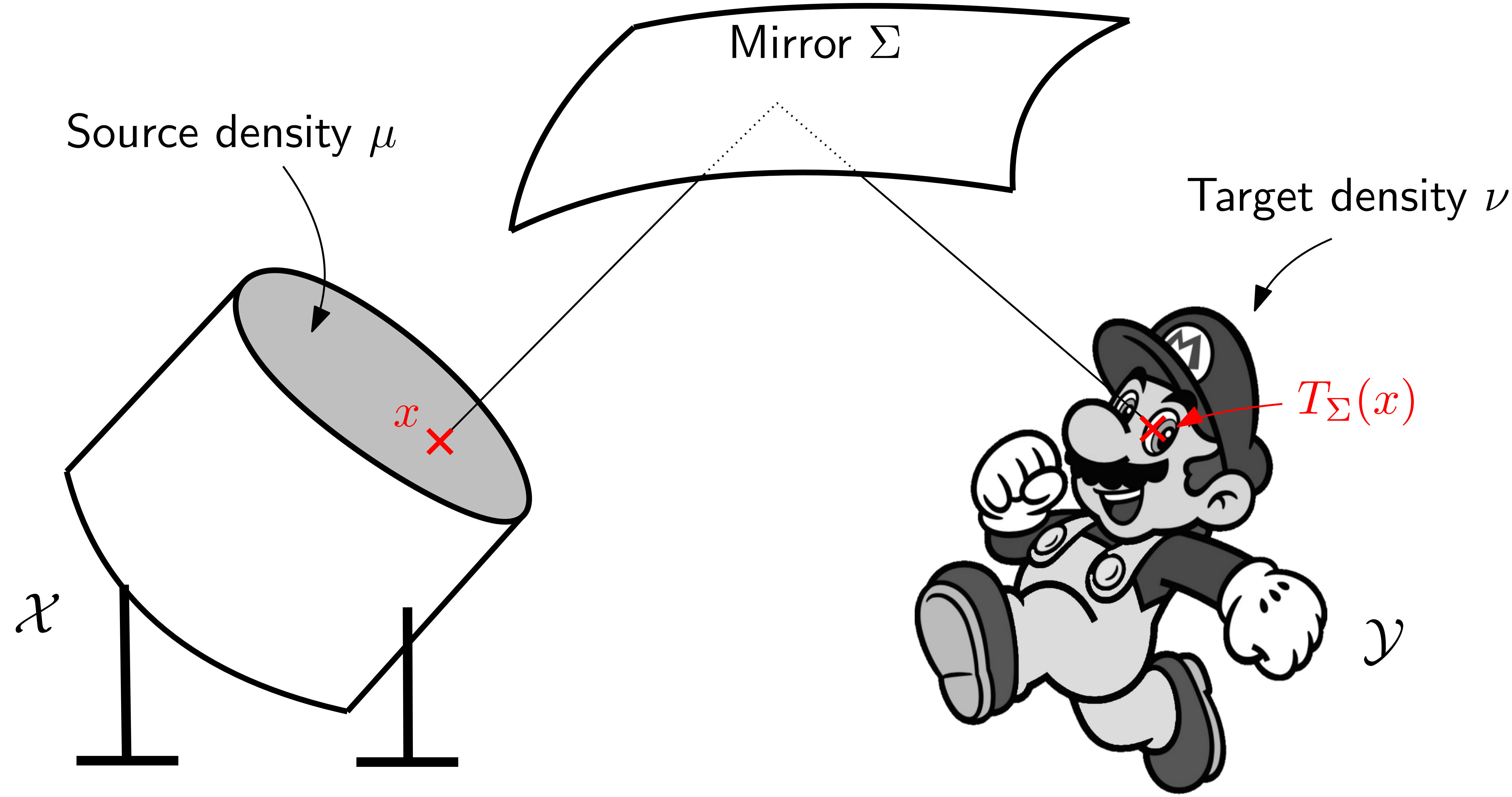
Goal: Construct a mirror that reflects a given light source toward a prescribed target.

Non-imaging optics: transport of measure



Input: Light source $\mathcal{X}$ with intensity $\mu \in \mathcal{P}(\mathcal{X})$.
Target distribution $\mathcal{Y}$ with intensity $\nu \in \mathcal{P}(\mathcal{Y})$.

Non-imaging optics: transport of measure



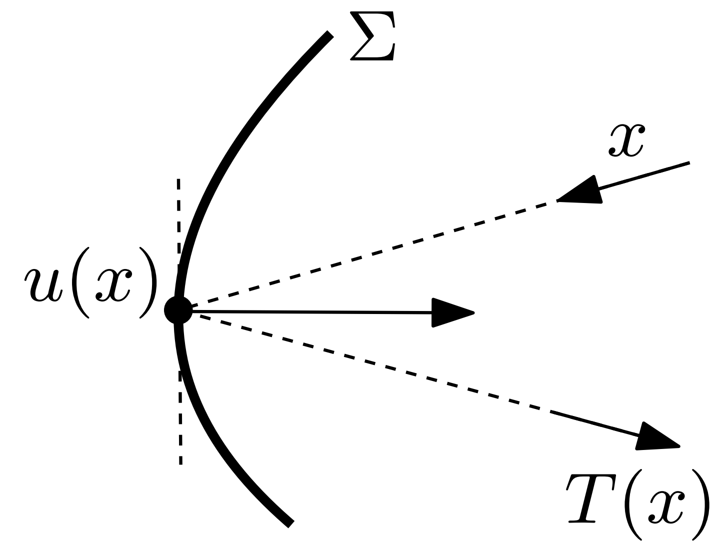
Input: Light source $\mathcal{X}$ with intensity $\mu \in \mathcal{P}(\mathcal{X})$.
Target distribution $\mathcal{Y}$ with intensity $\nu \in \mathcal{P}(\mathcal{Y})$.

Output: A surface Σ such that $T_{\Sigma\#}\mu = \nu$ with $T_{\Sigma\#}\mu(B) = \mu(T_\Sigma^{-1}(B))$

Optimal transport & Generated Jacobian eq.

- Assume $\mu(x) = \rho(x)\,\mathrm{d}\,x$ and $\nu(y) = \sigma(y)\,\mathrm{d}\,y$
  then $T_{\#}\mu = \nu$ amounts to: $\forall x \in \mathcal{X}, \sigma(T(x))\det(DT(x)) = \rho(x)$

- From Snell's law if Σ is parametrized by a function $u : \mathcal{X} \to \mathbb{R}$ then T is a function of x, $u(x)$ and $\nabla u(x)$.



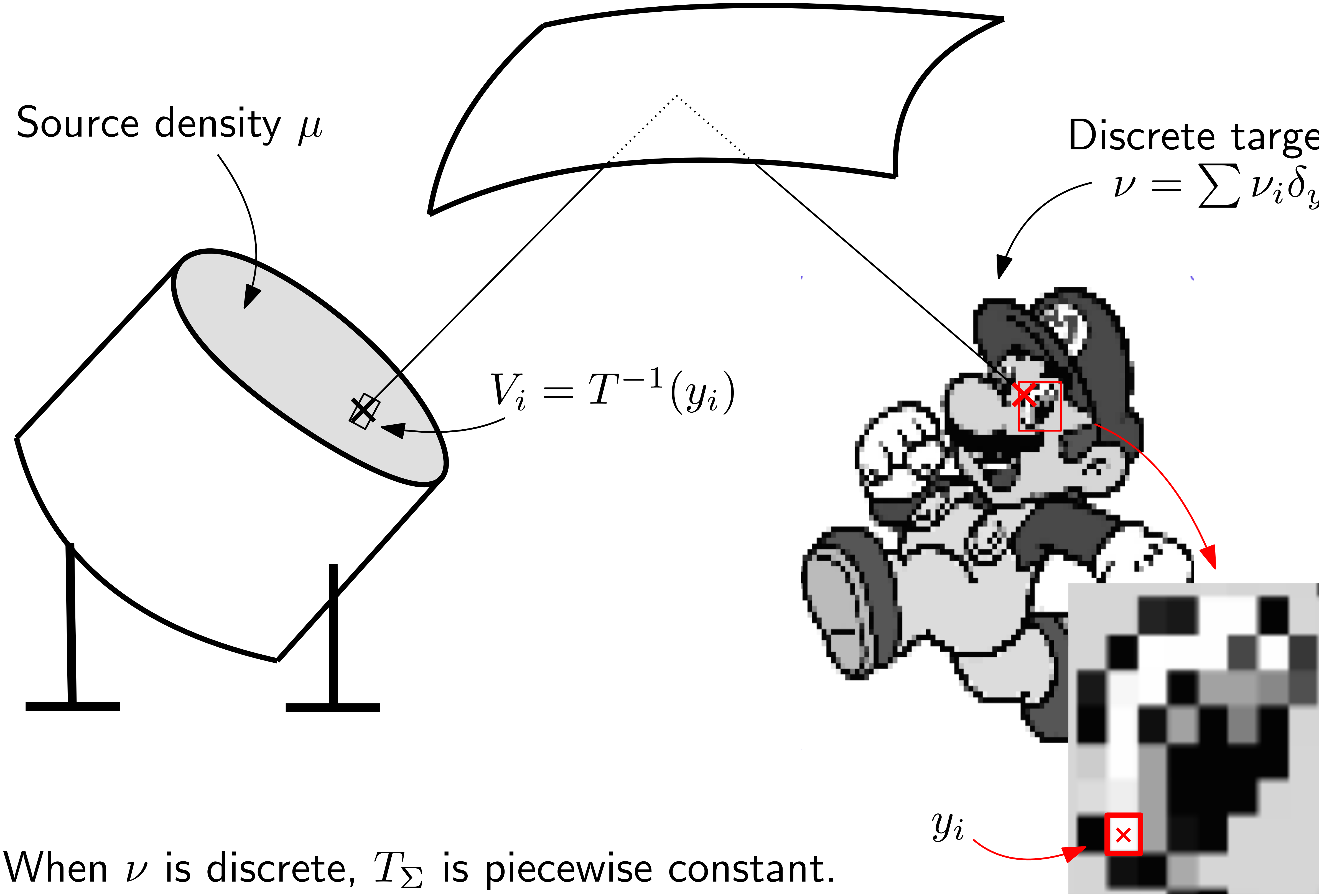
A Monge-Ampère type equation:

$$\forall x \in \mathcal{X}, \det(DT(x)) = \frac{\rho(x)}{\sigma(T(x))} \text{ with } T(x) = f(x, u(x), \nabla u(x)).$$

- In some particular cases this equation is OT, otherwise it is a GJE.

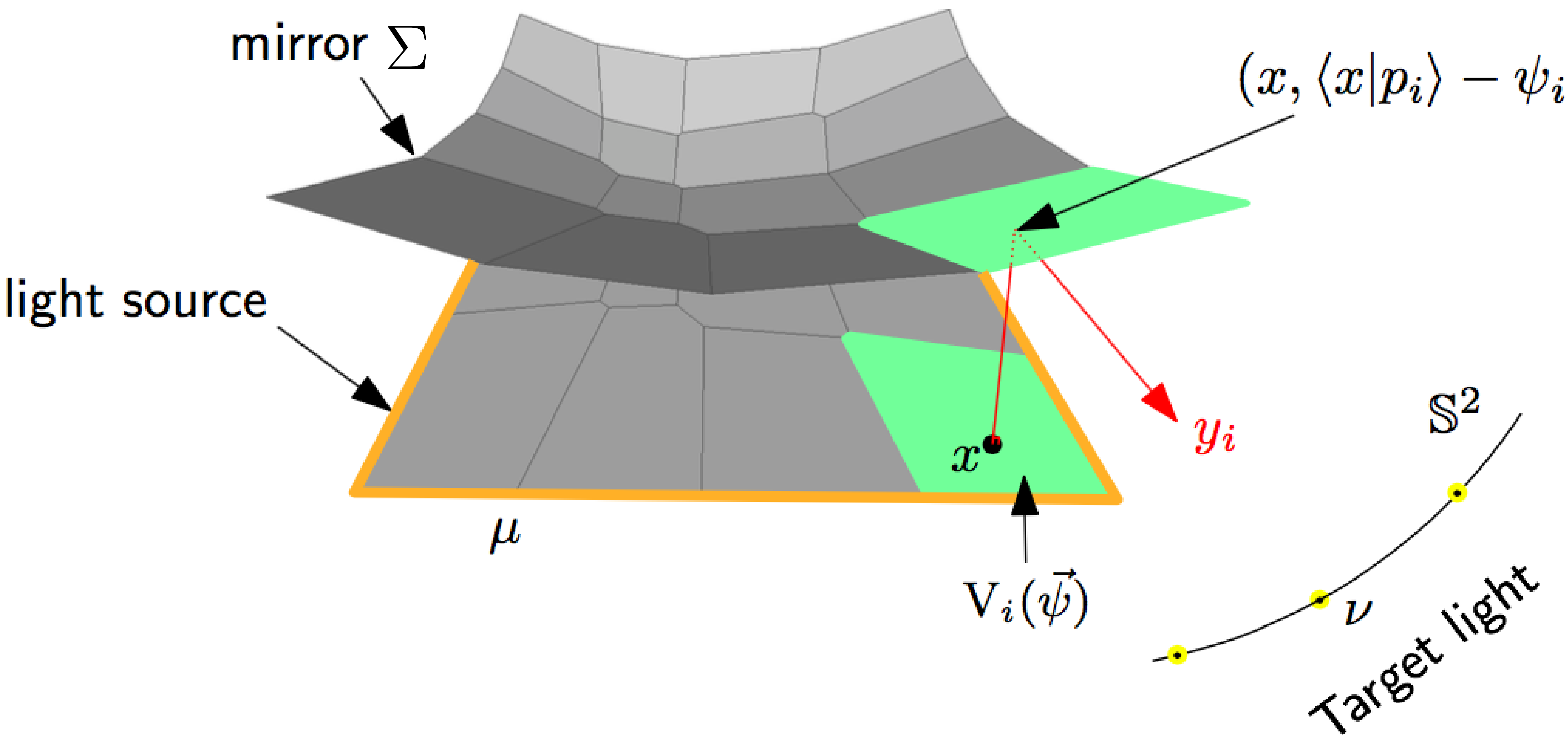
[Trudinger '14]

Discretization for numerical purposes



- When ν is discrete, T_Σ is piecewise constant.
- $T_{\Sigma\#}\mu = \nu$ becomes $\forall i, \mu(V_i) = \nu_i$ Prescribe the mass of each cell

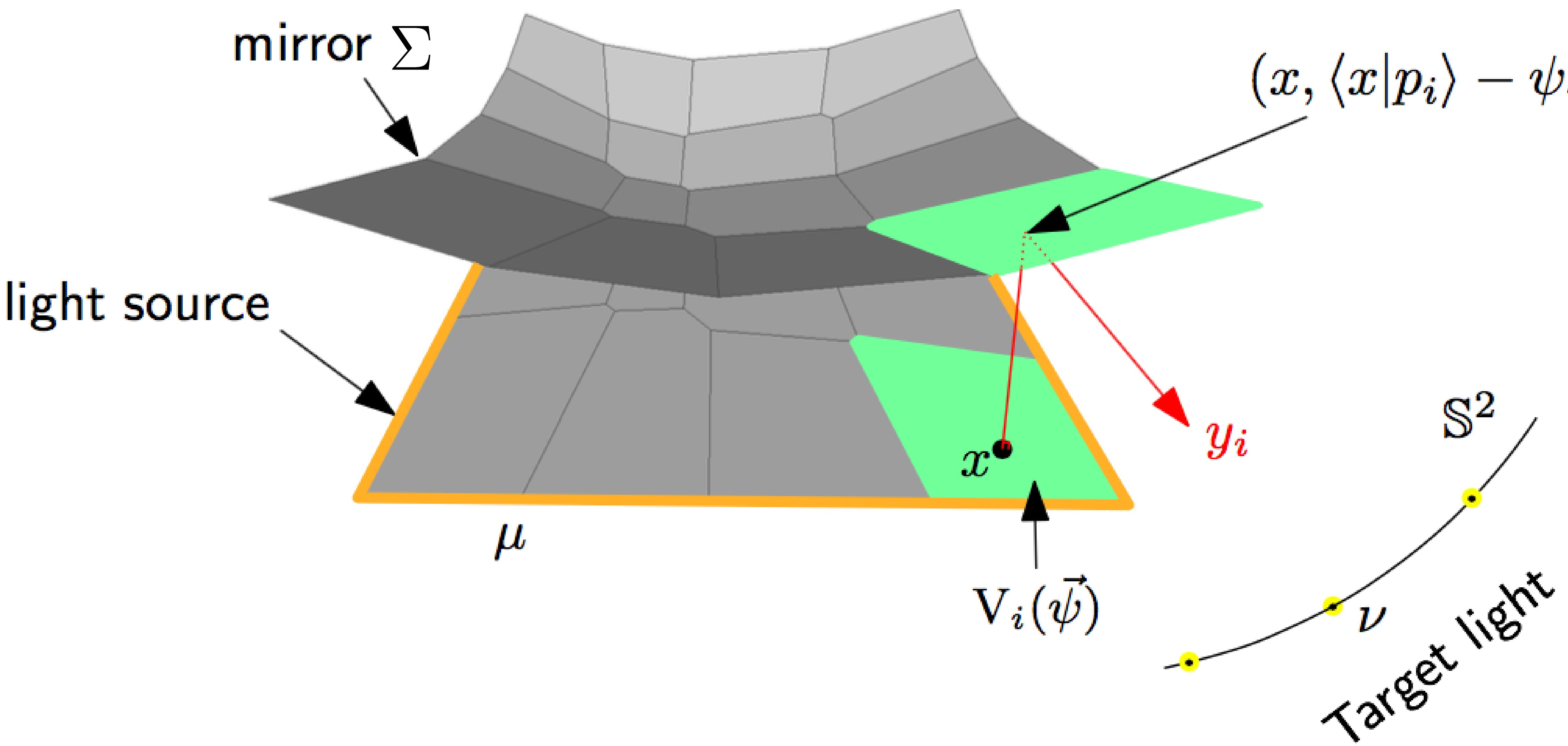
The far-field reflector problem



- Collimated light source $\mu \in \mathcal{P}^{\mathrm{ac}}(\mathcal{X})$.
- Far field target $\nu = \sum_{1 \leq i \leq N} \nu_i \delta_{y_i}$ with $y_i \in \mathbb{S}^2$.

Target "at infinity"

The far-field reflector problem



- Impose Σ to be the graph of $u(x) = \max_{1 \leq i N} \langle x|p_i\rangle - \psi_i$
- $V_i(\psi) = T^{-1}(y_i) = \{x \in \mathcal{X} \mid \forall j, \langle x|p_i\rangle - \psi_i \geq \langle x|p_j\rangle - \psi_j\}$

Problem: Find $\psi \in \mathbb{R}^N$ such that for all $i \in \{1, \cdots, N\}$, $\mu(V_i(\psi)) = \nu_i$

→ Semi-discrete Optimal Transport with $c(x, y) = -\langle x|y\rangle$

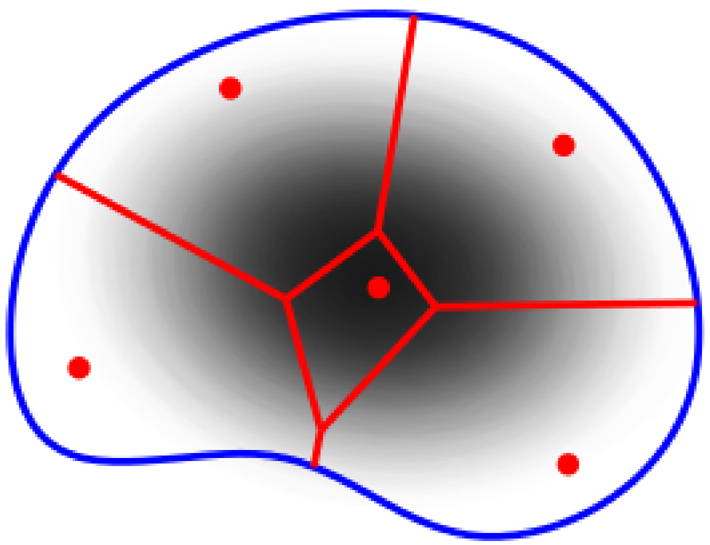
Semi-discrete optimal transport

Definition: (Laguerre diagram) Let $c : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be a twisted cost. For $\mathcal{Y} = (y_i)_{1 \leq i \leq N}$, and $\psi \in \mathbb{R}^{\mathcal{Y}}$, the Laguerre diagrams partitions $\mathcal{X}$ in N cells by

$$\mathrm{Lag}_y(\psi) = \{x \in \mathcal{X} \mid \forall z \neq y, c(x,y) + \psi(y) \leq c(x,z) + \psi(z)\}$$

Let T_ψ defined by $T_\psi(x) = y \iff x \in \mathrm{Lag}_y(\psi)$

Let $\mu \in \mathcal{P}(X)$ and $\nu = T_{\psi\#}\mu \in \mathcal{P}(\mathcal{Y})$



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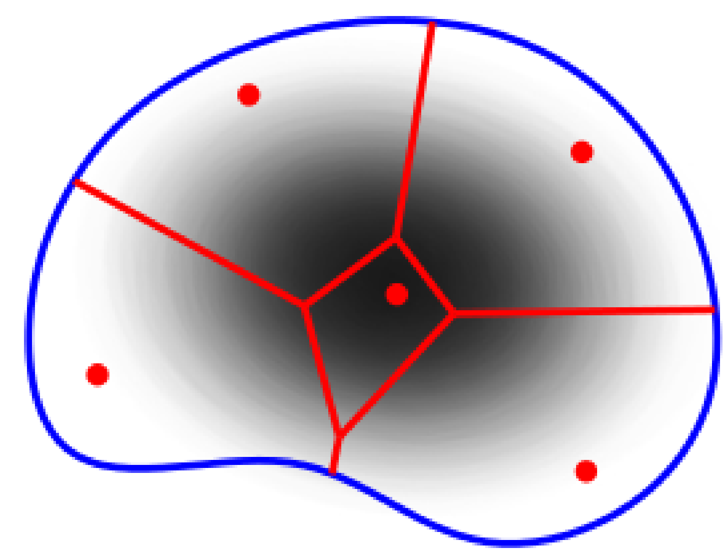

Let $\mu \in \mathcal{P}(X)$ and $\nu = T_{\psi\#}\mu \in \mathcal{P}(\mathcal{Y})$

Then $\forall (x,y) \in \mathcal{X} \times \mathcal{Y}$

$$c(x, T_\psi(x)) + \psi(T_\psi(x)) \leq c(x,y) + \psi(y)$$

So for any $\gamma \in \Pi(\mu,\nu)$, $\displaystyle\int c(x, T_\psi(x)) + \psi(T_\psi(x)) \,\mathrm{d}\,\mu \leq \int c(x,y) + \psi(y) \,\mathrm{d}\,\gamma$

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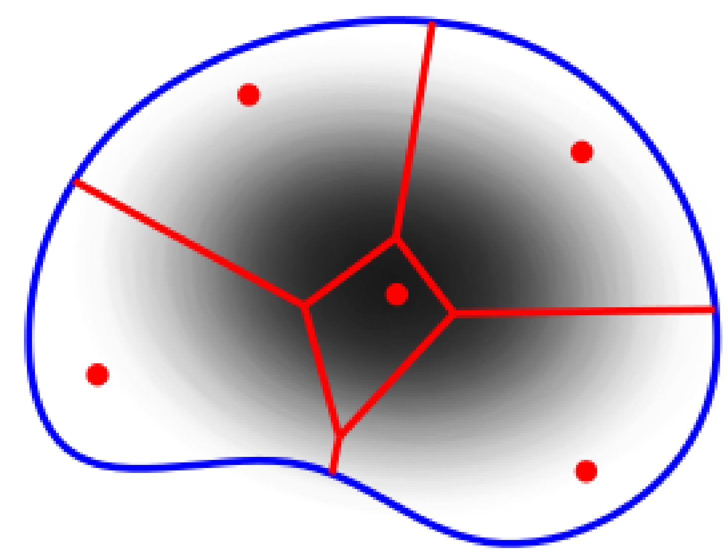

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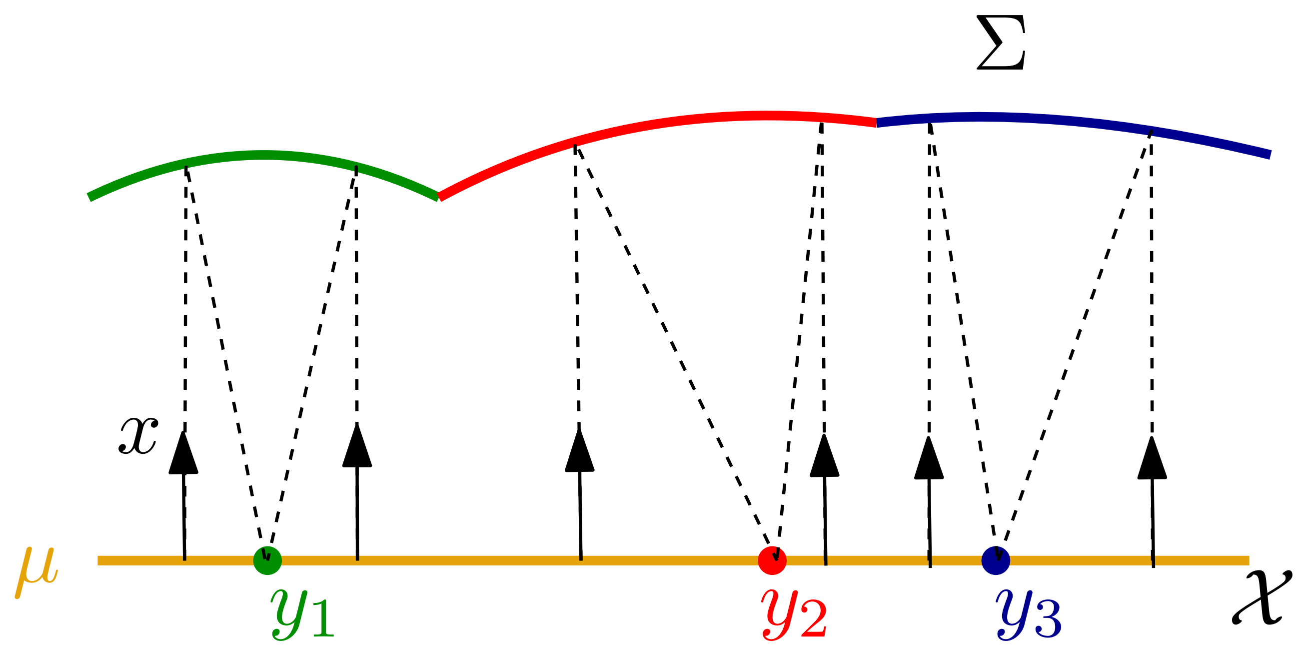
So for any $\gamma \in \Pi(\mu, \nu)$, $\int c(x, T_\psi(x)) + \cancel{\psi(T_\psi(x))} \, \mathrm{d}\,\mu \leq \int c(x,y) + \cancel{\psi(y)} \, \mathrm{d}\,\gamma$

$$\boxed{T_\psi \text{ is optimal between } \mu \text{ and } \nu = T_{\psi\#}\mu \text{ for the cost } c}$$

The Near-Field Parallel reflector



- Collimated light source:

  $\mu \in \mathcal{P}(\mathcal{X})$ abs. cont.

- Near field target:

  $$\nu = \sum_{1 \leq i \leq N} \nu_i \delta_{y_i} \text{ with } y_i \in \mathbb{R}^2.$$

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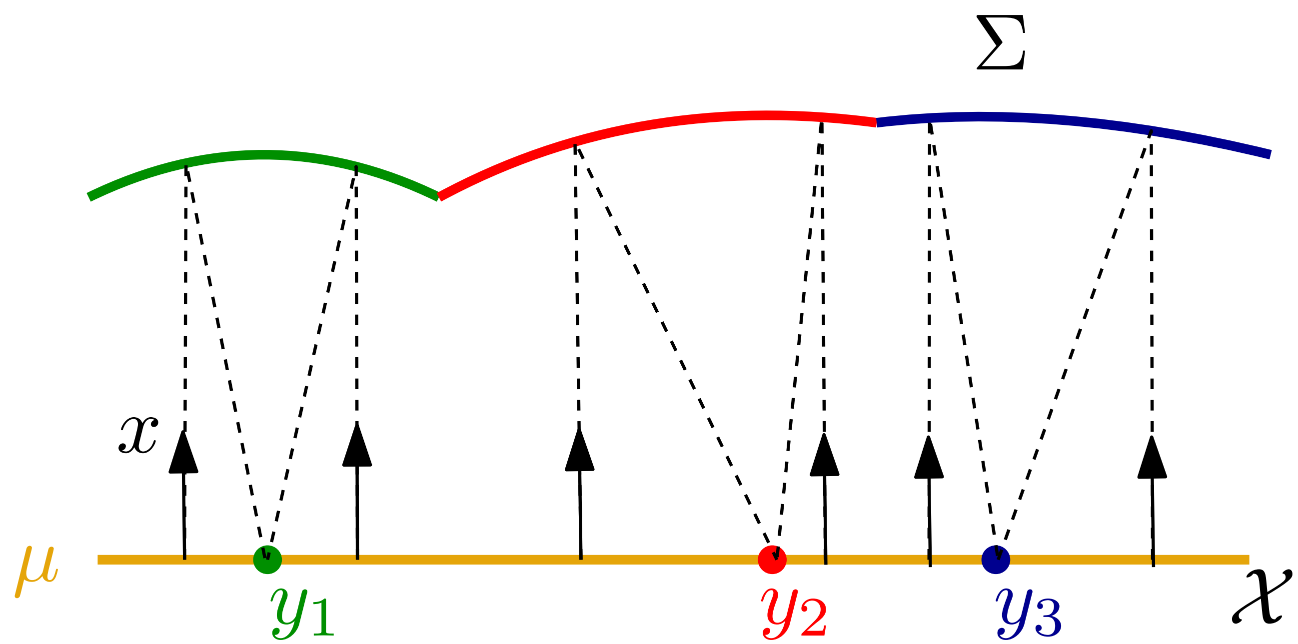

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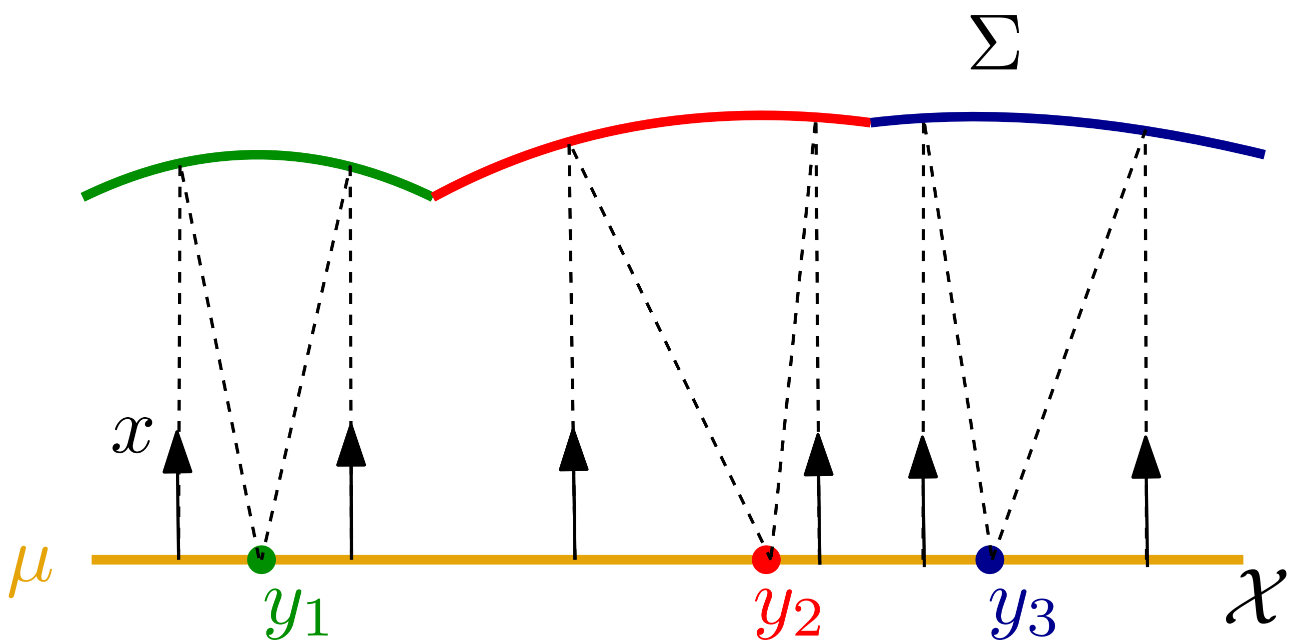
  $$\nu = \sum_{1 \le i \le N} \nu_i \delta_{y_i} \text{ with } y_i \in \mathbb{R}^2.$$
- Mirror Σ is a maximum of paraboloids of focus (y_i) .

- Parametrization of Σ: $u(x) = \max_{1 \le i \le N} \frac{1}{2\psi_i} - \frac{\psi_i}{2} \|x - y_i\|^2$

The Near-Field Parallel reflector



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$$\mathrm{Lag}_i(\psi) = \left\{ x \in \mathcal{X} \mid \forall j, \frac{1}{2\psi_i} - \frac{\psi_i}{2}\|x - y_i\|^2 \geq \frac{1}{2\psi_j} - \frac{\psi_j}{2}\|x - y_j\|^2 \right\}$$

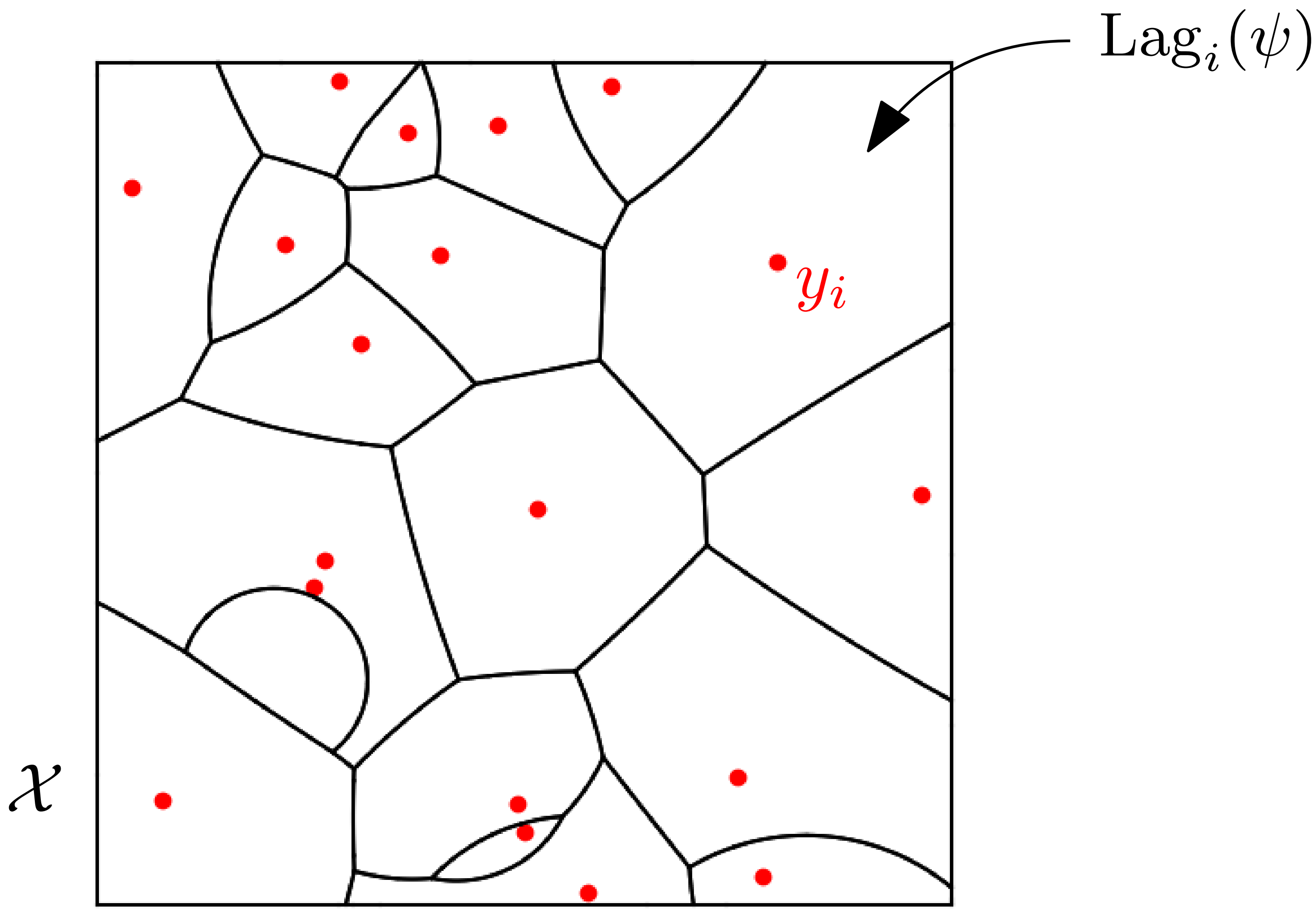
Problem: Find $\psi \in \mathbb{R}^N$ such that for all $i \in \{1, \cdots, N\}$, $\mu(\mathrm{Lag}_i(\psi)) = \nu_i$

Not linear in $\psi \rightarrow$ not optimal transport.

Semi-discrete generated Jacobian eq.

Definition: (Generating function & generalized Laguerre cells)

- **Generating function** $G : \mathcal{X} \times \mathcal{Y} \times \mathbb{R} \to \mathbb{R}$ satisfies (Reg), (Twist), (UC) and (Mono).
- **Generalized Laguerre diagram** for $\psi \in \mathbb{R}^N$
  $\mathrm{Lag}_i(\psi) = \{x \in \mathcal{X} \mid \forall j \in \{1, \cdots, N\}, G(x, y_i, \psi_i) \geq G(x, y_j, \psi_j)\}$



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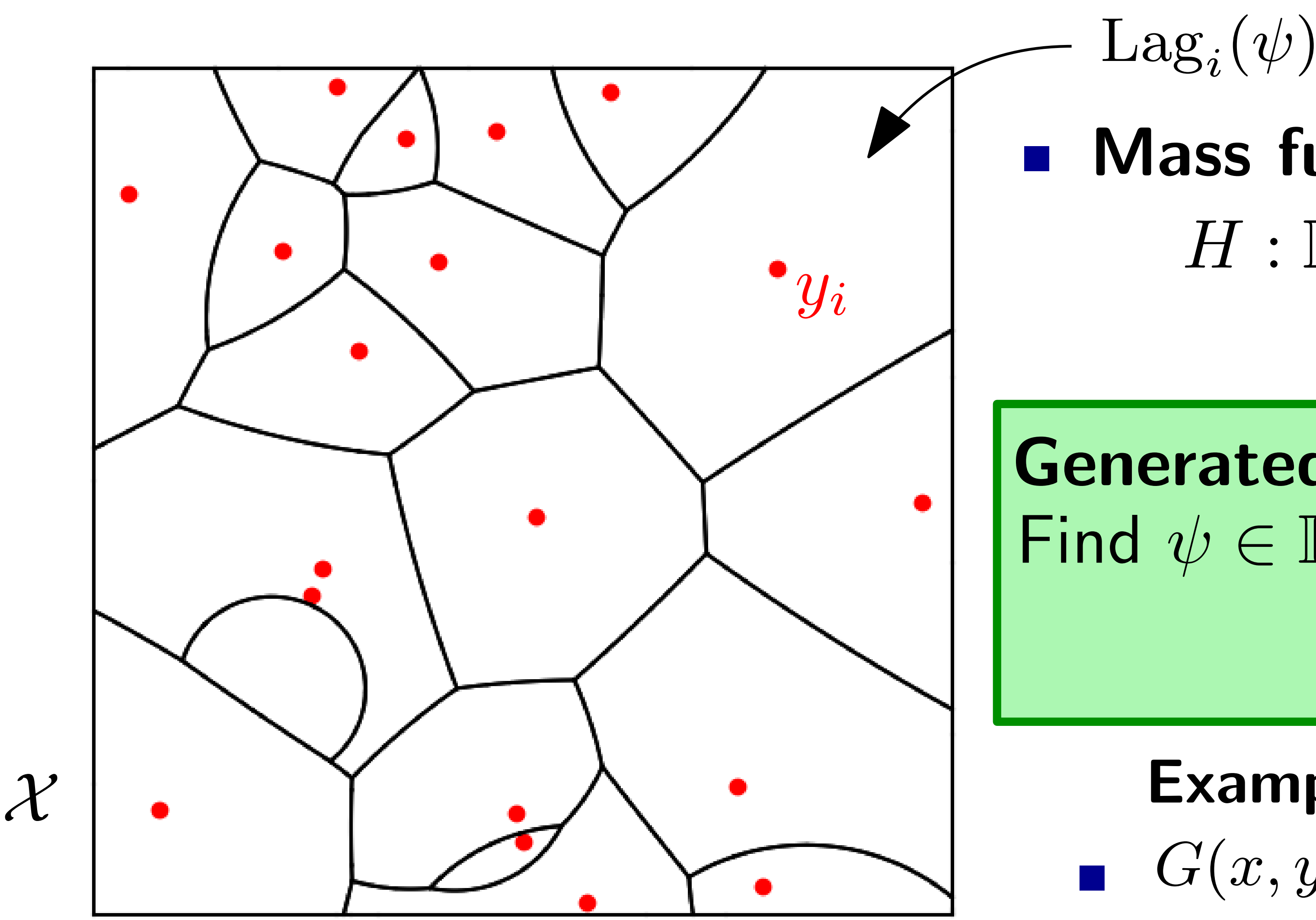

- **Mass function:**

$$\begin{aligned} H : \mathbb{R}^N &\to \mathbb{R}^N \\ \psi &\mapsto \left(\mu(\mathrm{Lag}_i(\psi))\right)_{1 \leq i \leq N} \end{aligned}$$

Generated Jacobian eq: *(Trudinger '14)*
Find $\psi \in \mathbb{R}^N$ such that

$$H(\psi) = \nu$$

Examples:

- $G(x, y, v) = -c(x, y) - v \qquad (OT)$
- $G(x, y, v) = \frac{1}{2v} - \frac{v}{2}\|x - y\|^2 \qquad (NFPar)$

Semi-discrete generated Jacobian eq.

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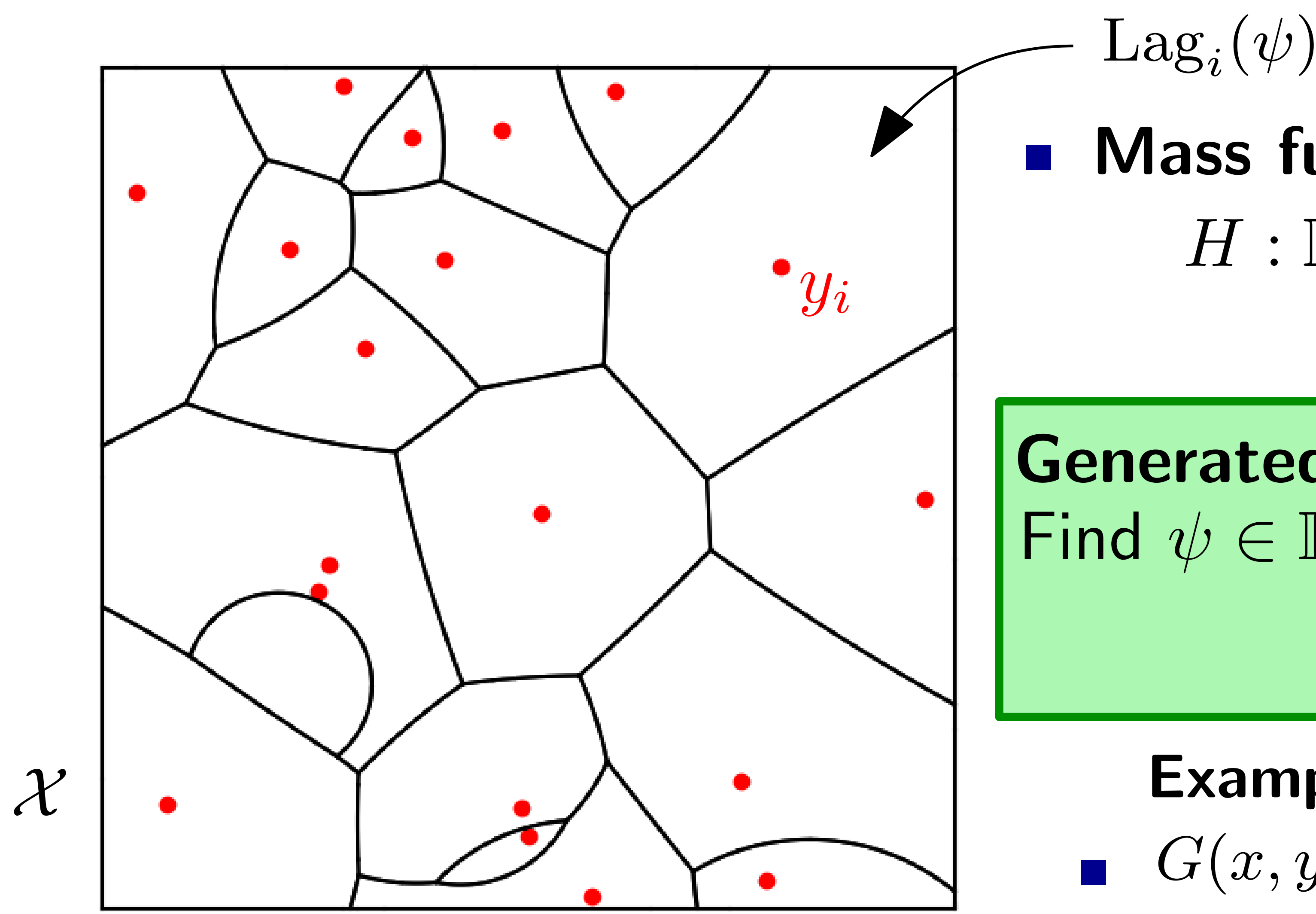

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Find $\psi \in \mathbb{R}^N$ such that

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Solve using Newton alg.

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- $G(x, y, v) = -c(x, y) - v$ $\quad (OT)$
- $G(x, y, v) = \frac{1}{2v} - \frac{v}{2}\|x - y\|^2$ $\quad (NFPar)$

The optimal transport case

Dual formulation of Optimal Transport:

$$\max_{\psi \in \mathbb{R}^{\mathcal{Y}}} \int_{\mathcal{X}} \psi^c \,\mathrm{d}\,\mu - \sum_{y \in \mathcal{Y}} \psi(y)\nu(y) := \mathcal{K}(\psi)$$

where $\psi^c(x) = \min_i c(x, y_i) + \psi_i$

- $\mathcal{K}$ is concave and invariant by addition of a constant.
- $\mathrm{Lag}_i(\psi) = \{x \mid \forall j, c(x, y_i) + \psi_i \leq c(x, y_j) + \psi_j\}$
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Theorem: *[Mérigot-Thibert '16]* If the cost is $\mathcal{C}^2$ and twisted, and $\mathrm{spt}(\mu)$ is connected and compact, then $\mathcal{K}$ **is $\mathcal{C}^2$ and locally strongly concave** on $\mathcal{S}^+ \cap \mathbf{1}^\perp$, where $\mathcal{S}^+ = \{x \mid \forall i, H_i(\psi) > 0\}$.

- In particular DH is symmetric non-positive definite
- Solving $H(\psi) = \nu$ amounts to maximize a (strongly) concave function.

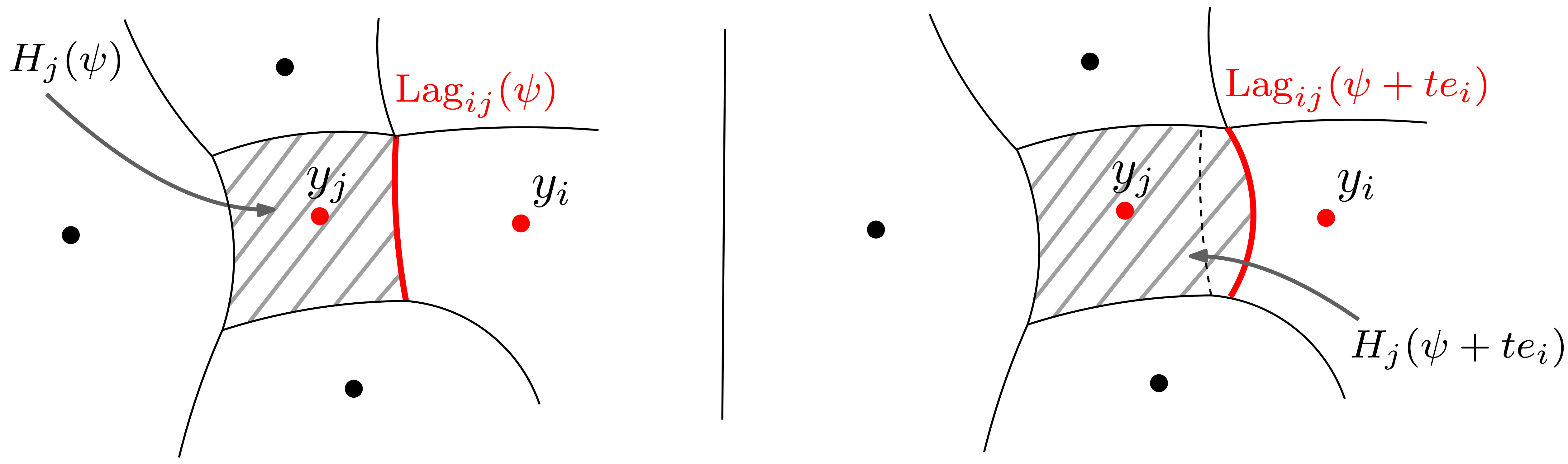
Differential of the mass function H

Proposition 1:*[G-M-T '21]*(Formula for DH) Assume that G is $\mathcal{C}^2$ and $\mathrm{spt}(\mu)$ is compact, then for any $i \neq j$ we have:

- $$\frac{\partial H_j}{\partial \psi_i}(\psi) = \int_{\mathrm{Lag}_{ij}(\psi)} \rho(x) \frac{|\partial_3 G(x, y_i, \psi_i)|}{\|\nabla_x G(x, y_j, \psi_j) - \nabla_x G(x, y_i, \psi_i)\|} d\mathcal{H}^{d-1}(x) \geq 0$$

$\mathrm{Lag}_{ij} = \mathrm{Lag}_i \cap \mathrm{Lag}_j$

- $$\sum_i H_i(\psi) = 1 \implies \frac{\partial H_i}{\partial \psi_i}(\psi) = -\sum_{j \neq i} \frac{\partial H_j}{\partial \psi_i}(\psi)$$



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> 0 by (Mono)

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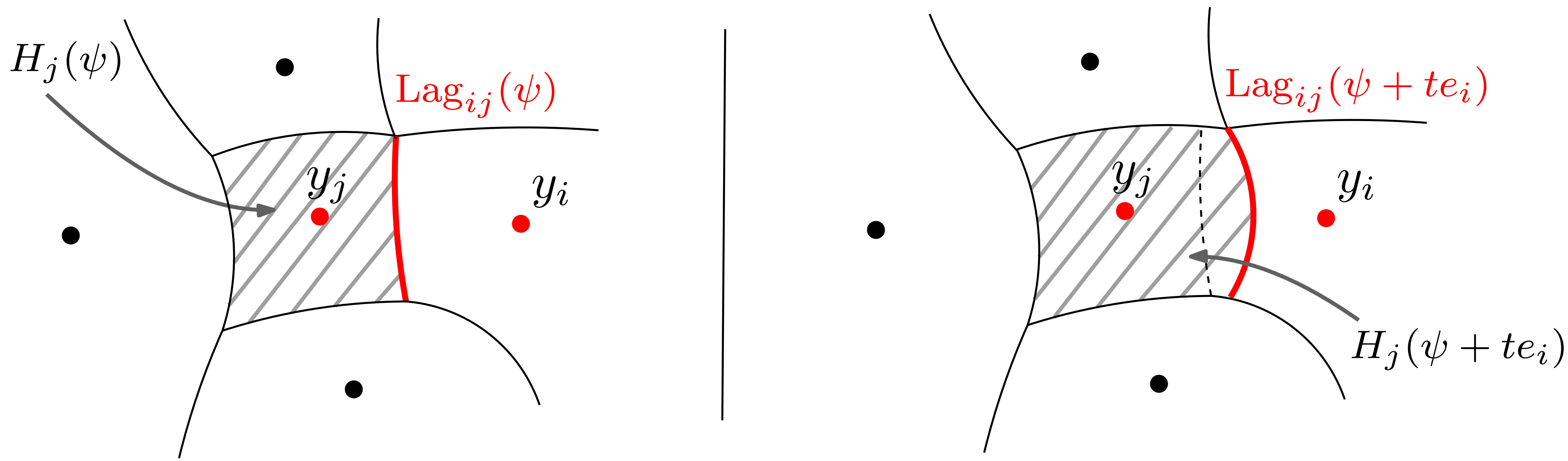
$\mathrm{Lag}_{ij} = \mathrm{Lag}_i \cap \mathrm{Lag}_j$

$\neq 0$ by (Twist)

- $$\sum_i H_i(\psi) = 1 \implies \frac{\partial H_i}{\partial \psi_i}(\psi) = -\sum_{j \neq i} \frac{\partial H_j}{\partial \psi_i}(\psi)$$ < 0 if $H_i(\psi) > 0$



Descent direction for Newton

Proposition 2: *[G-M-T '21]* Let $\psi \in \mathcal{S}^+ = \{\psi \in \mathbb{R}^N \mid \forall i, H_i(\psi) > 0\}$, then

- The differential $DH(\psi)$ is of rank $N-1$.
- Its image is $\mathrm{Im}(DH(\psi)) = \mathbf{1}^\perp$ where $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^N$.
- Its kernel is $\ker(DH(\psi)) = \mathrm{span}(w)$ with $w_i > 0$ for $1 \leq i \leq N$.

- Consequence of Perron-Frobenius for irreducible matrices
- The differential DH has no reason to be symmetric

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Corollary: (Descent direction) Let $\psi \in \mathcal{S}^+$, the system

$$\begin{cases} DH(\psi)u = H(\psi) - \nu \\ u_1 = 0 \end{cases}$$

has a unique solution $u \in \mathbb{R}^N$.

Idea of proof:

- We have $H(\psi) - \nu \in \mathbf{1}^\perp = \mathrm{Im}(DH(\psi))$.
- Fixing $u_1 = 0$ is possible because of the structure of $\ker(DH(\psi))$.
- Uniqueness comes from the rank of $DH(\psi)$.

Damped Newton algorithm

Newton algorithm for solving $H(\psi) = \nu$

Initialize $\psi^0 \in \mathcal{S}^\delta = \{\psi \in \mathbb{R}^N \mid \forall i, H_i(\psi) > \delta\}$ and $\varepsilon > 0$.

While $\|H(\psi) - \nu\| \geq \varepsilon$:

ψ_1^k is fixed

$\longrightarrow$ Compute u^k solution of $\begin{cases} DH(\psi)u^k = H(\psi) - \nu \\ u_1^k = 0 \end{cases}$

$\longrightarrow$ Define for $\tau \in [0,1]$, $\psi^{k,\tau} = \psi^k - \tau u^k$.

Damping Parameter

$\longrightarrow$ Compute $\tau^k = \sup\left\{\tau \in [0,1] \;\middle|\; \begin{array}{r} \|H(\psi^{k,\tau}) - \nu\| \leq (1 - \frac{\tau}{2})\|H(\psi^k) - \nu\| \\ \text{and } \psi^{k,\tau} \in \mathcal{S}^\delta \end{array}\right\}$

$\longrightarrow$ Put $\psi^{k+1} \leftarrow \psi^{k,\tau^k}$ and $k \leftarrow k+1$

Iterate stays in admissible set

Return ψ^k.

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Linear convergence

- $\longrightarrow$ Define for $\tau \in [0,1]$, $\psi^{k,\tau} = \psi^k - \tau u^k$.

Damping Parameter

- $\longrightarrow$ Compute $\tau^k = \sup\left\{\tau \in [0,1] \;\middle|\; \|H(\psi^{k,\tau}) - \nu\| \leq (1 - \frac{\tau}{2})\|H(\psi^k) - \nu\| \text{ and } \psi^{k,\tau} \in \mathcal{S}^\delta\right\}$

Iterate stays in admissible set

- $\longrightarrow$ Put $\psi^{k+1} \leftarrow \psi^{k,\tau^k}$ and $k \leftarrow k+1$

Return ψ^k.

Theorem : *[G-M-T '21]* (Convergence) Assume that the support of μ is connected and compact and that the set $\mathcal{Y}$ is generic. Then $\exists \tau^* > 0$ s.t

$$\|H(\psi^k) - \nu\| \leq \left(1 - \frac{\tau^*}{2}\right)^k \|H(\psi^0) - \nu\|$$

Proof: Bound τ^k below for any k by compactness of the set

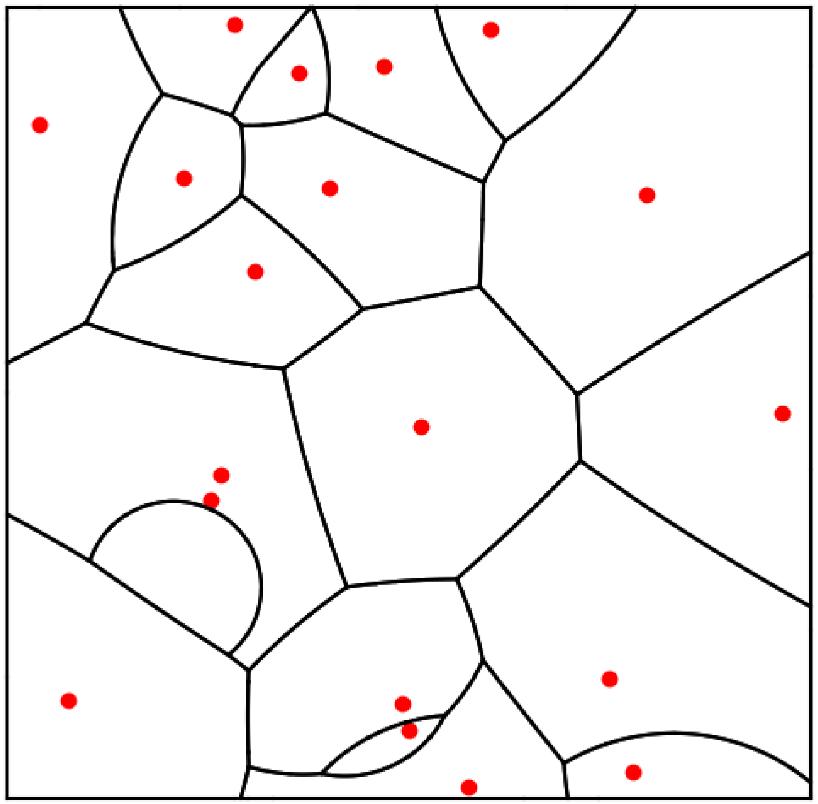
$$K = \{\psi \in \mathcal{S}^\delta \mid \psi_1 = \psi_1^0 \text{ and } \|H(\psi) - \nu\| \leq \|H(\psi^0) - \nu\|\}.$$

Implementation for the Near field reflector

- Laguerre diagram for (NF-par):

$$\mathrm{Lag}_i(\psi) = \left\{ x \in \mathbb{S}^2 \mid \forall j, \frac{1}{2\psi_i} - \frac{\psi_i}{2}\|x-y\|^2 \geq \frac{1}{2\psi_j} - \frac{\psi_j}{2}\|x-y\|^2 \right\}$$

- Particular case of Möbius diagram



Definition: (Power and Möbius diagram)

- Power diagram : $\mathrm{Pow}_i(c,r) = \{x \mid \forall j, \|x-c_i\|^2 + r_i \leq \|x-c_j\|^2 + r_j\}$
- Möbius diagram : $\mathrm{Mob}_i(y,\lambda,\mu) = \{x \mid \forall j, \lambda_i\|x-y_i\|^2 + \mu_i \leq \lambda_j\|x-y_j\|^2 + \mu_j\}$

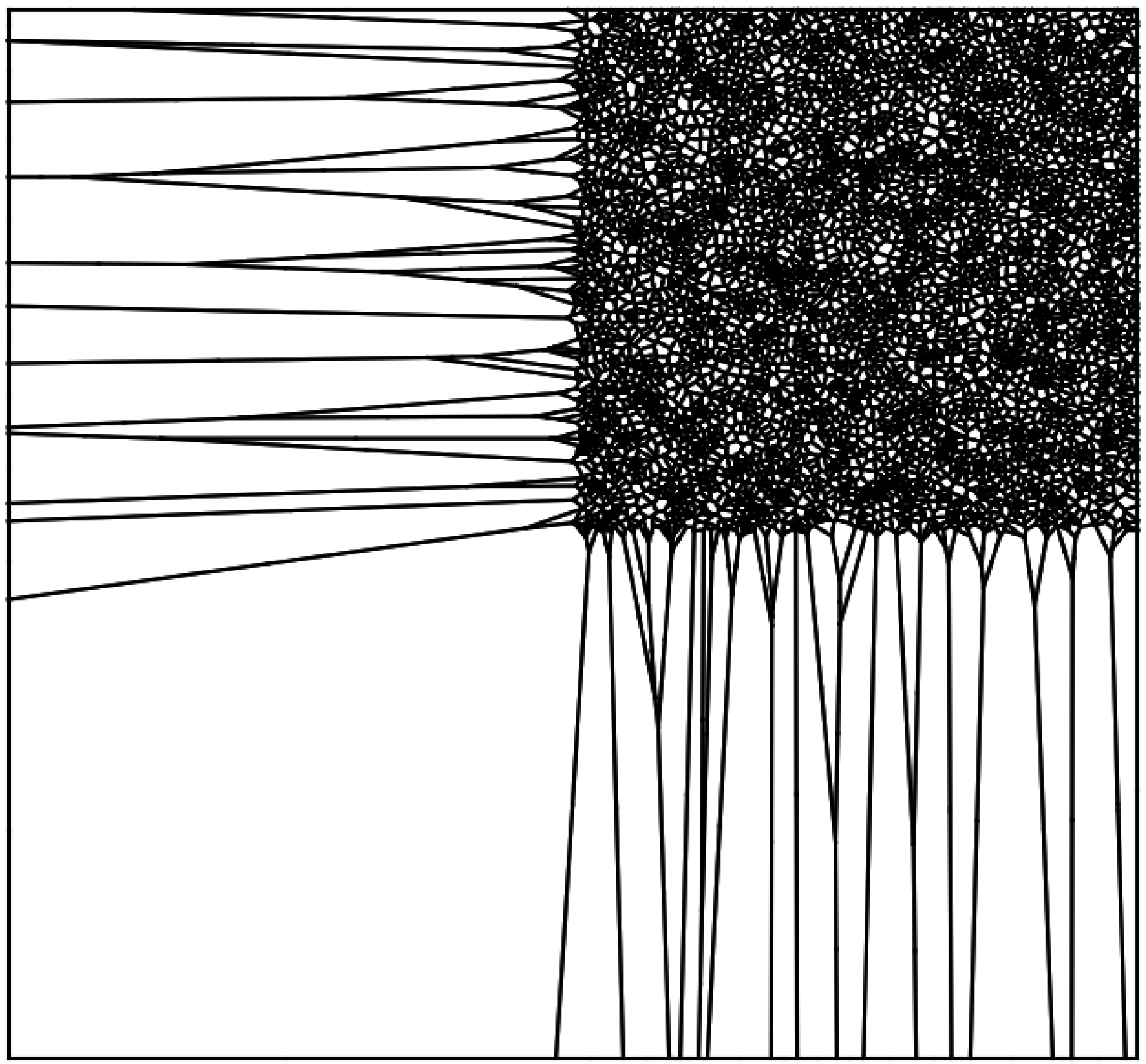
Theorem: *[Boissonnat-Wormser-Yvinec '07]*. For any family $(\lambda_i, \mu_i)_i \subset \mathbb{R}$, and $(y_i)_i \subset \mathbb{R}^d$ there exists $(r_i)_i \subset \mathbb{R}$ and $(c_i)_i \subset \mathbb{R}^{d+1}$ such that

$$\mathrm{Mob}_i(y,\lambda,\mu) = \Pi(\mathrm{Pow}_i(c,r) \cap P)$$

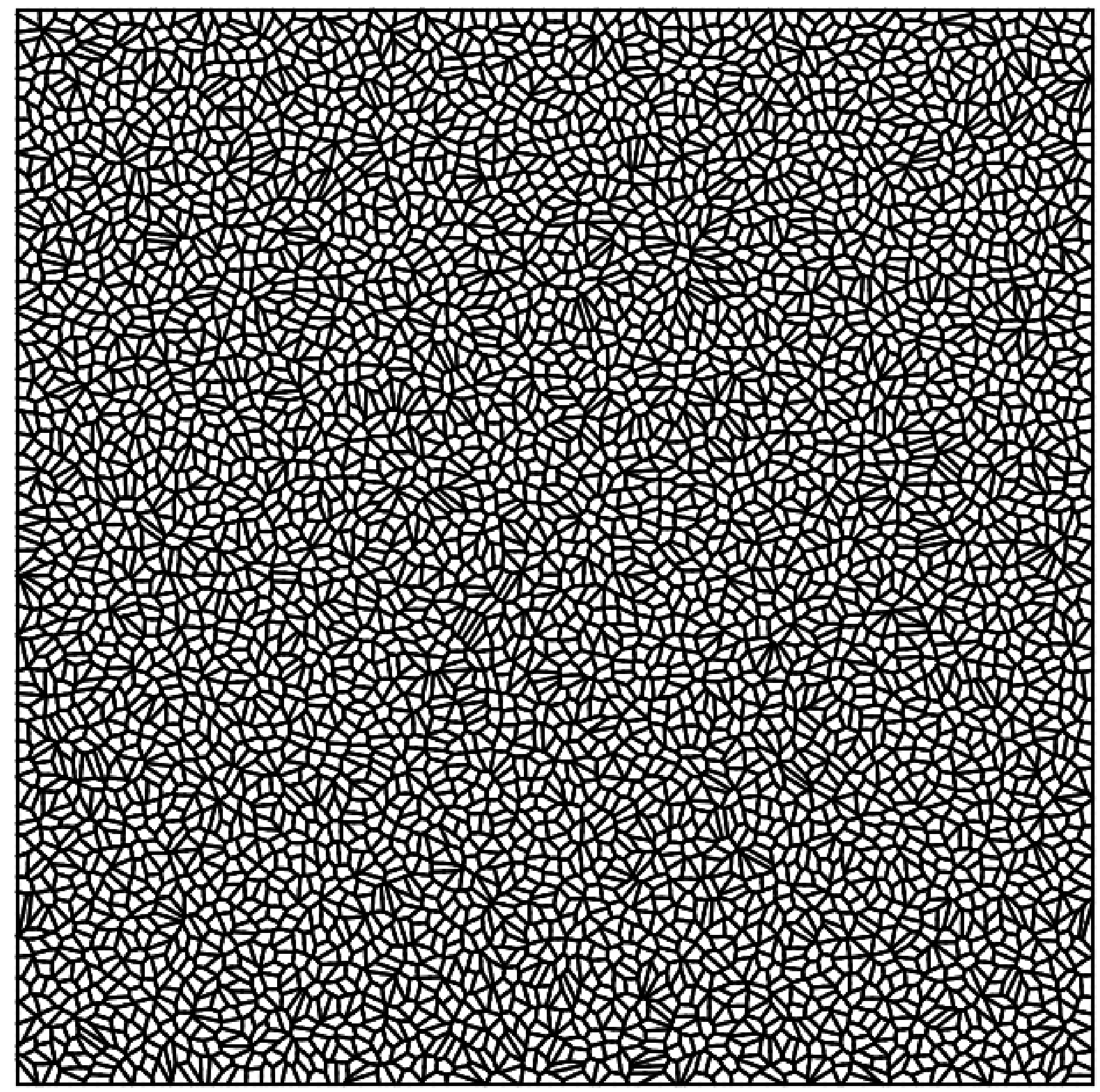
where $P = \{(x, \|x\|^2), x \in \mathbb{R}^d\}$ and Π is the orthogonal projection of $\mathbb{R}^{d+1}$ on $\mathbb{R}^d \times \{0\}$.

Numerical experiments

- $\mathcal{X} = [-1, 1]^2$ with μ uniform
- $\mathcal{Y} \subset [0, 1]^2$, with ν uniform and $N = 5000$.



Initial diagram

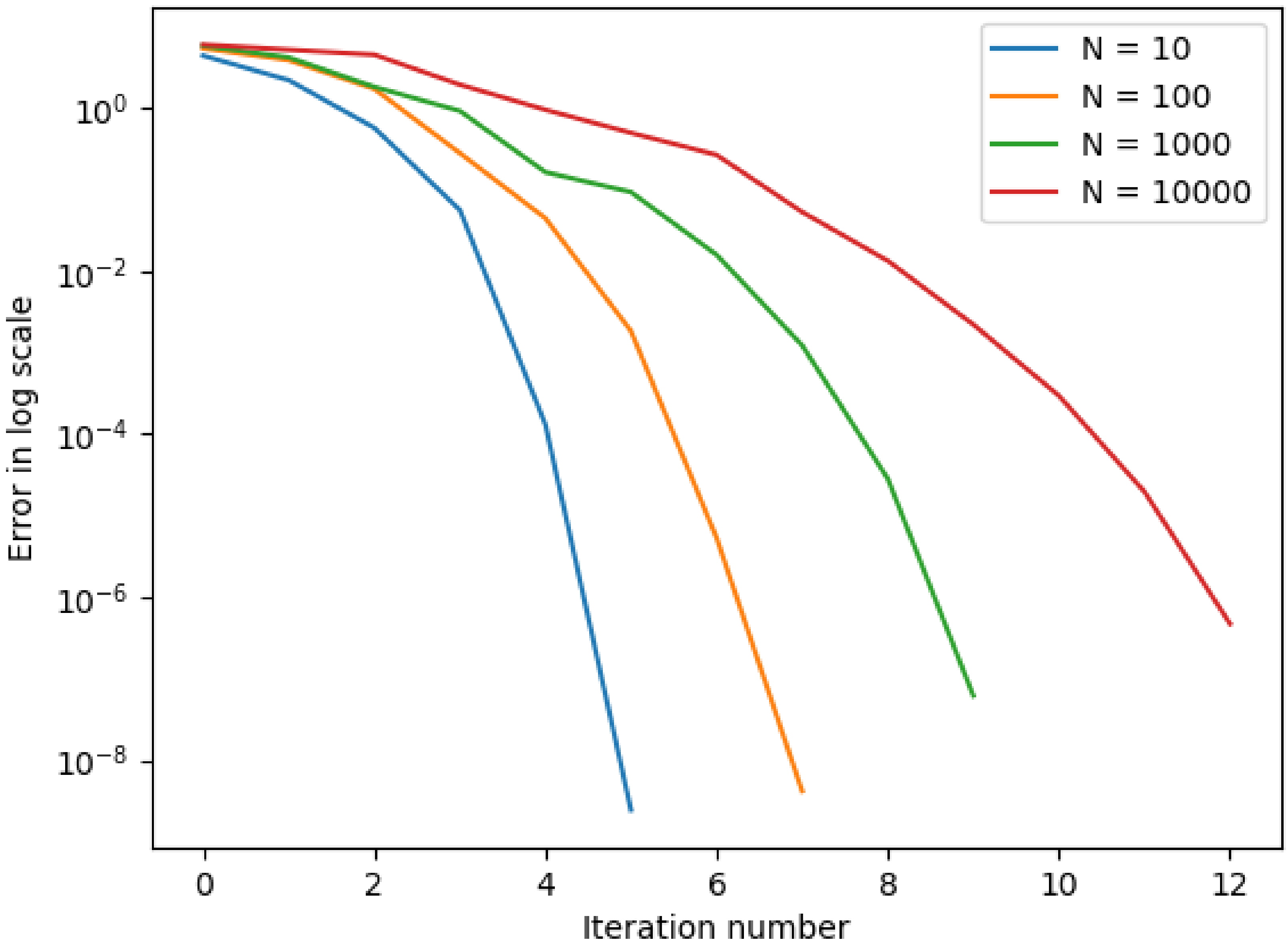


Final diagram

Laguerre diagram before and after convergence of the Newton algorithm

Numerical experiments

- $\mathcal{X} = [-1, 1]^2$ with μ uniform
- $\mathcal{Y} \subset [0, 1]^2$, with ν uniform



Convergence rate for different values of N.

Thank you for your attention

Appendix

Generating Function:

- $G(x,y,v)$ is $\mathcal{C}^1$ in x and v and $\sup_{\mathcal{X}\times\mathcal{Y}\times[\alpha,\beta]} |\nabla_x G(x,y,v)| < +\infty$ (Reg)

- $\forall (x,y,v) : \partial_v G(x,y,v) < 0$ (Mono)

- $\forall x \in \mathcal{X}, (y,v) \mapsto (G(x,y,v), \nabla_x G(x,y,v))$ is injective on $\mathcal{Y} \times \mathbb{R}$ (Twist)

- $\forall y \in Y, \lim_{v \to -\infty} \inf_{x \in \mathcal{X}} G(x,y,v) = +\infty$ (UC)

A stochastic algorithm for GJE

Entropic regularization:

- Regularized cells: $\mathcal{L}_{\varepsilon,i}[\psi](x) = \dfrac{e^{G(x,y_i,\psi_i)/\varepsilon}}{\sum_k e^{G(x,y_k,\psi_k)/\varepsilon}} \xrightarrow[\varepsilon \to 0]{} \begin{cases} 1 \text{ if } x \in \mathrm{Lag}_i(\psi) \\ 0 \text{ otherwise} \end{cases}$

- Regularized mass function: $H_i^\varepsilon(\psi) = \displaystyle\int_X \mathcal{L}_{\varepsilon,i}[\psi](x)\,\mathrm{d}\,\mu(x) \xrightarrow[\varepsilon \to 0]{} H_i(\psi)$

Regularized GJE: Find $\psi \in \mathbb{R}^N$ such that $H^\varepsilon(\psi) = \nu$

A stochastic algorithm for GJE

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Regularized GJE: Find $\psi \in \mathbb{R}^N$ such that $H^\varepsilon(\psi) = \nu$

Fixed point iterate: $\psi^{k+1} = \psi^k + \tau^k(H^\varepsilon(\psi) - \nu)$

A stochastic algorithm for GJE

Entropic regularization:

- Regularized cells: $\mathcal{L}_{\varepsilon,i}[\psi](x) = \dfrac{e^{G(x,y_i,\psi_i)/\varepsilon}}{\sum_k e^{G(x,y_k,\psi_k)/\varepsilon}} \xrightarrow[\varepsilon\to 0]{} \begin{cases} 1 \text{ if } x \in \mathrm{Lag}_i(\psi) \\ 0 \text{ otherwise} \end{cases}$

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Regularized GJE: Find $\psi \in \mathbb{R}^N$ such that $H^\varepsilon(\psi) = \nu$

Stochastic fixed point iterate:

$$\psi^{k+1} = \psi^k + \tau^k(\mathcal{L}_\varepsilon[\psi](x_k) - \nu)$$

where $x_k \sim \mu$ so that $\mathbb{E}(\mathcal{L}_\varepsilon[\psi](x_k)) = H^\varepsilon(\psi)$

- Stochastic gradient descent in the case of optimal transport.
- Numerical experiments converge for $\tau^k = \frac{1}{\sqrt{k}}$
- Proof of convergence is an open problem.